COMPUTATIONAL METHODS FOR FRACTIONAL DIFFERENTIAL EQUATION ARISING IN BIOLOGICAL PROBLEMS

Research Protocol

For Registration in Ph.D. Program

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CERTIFICATE

This is to certify that this research protocol entitled "**Computational Methods for Fractional Differential Equation Arising in Biological Problems"** by **RASHMI AGRAWAL**, submitted in fulfilment of the requirement for registration in Ph.D. Program in Department of Mathematics under Faculty of Science of JECRC University, Jaipur, during the academic year 2019-2020, is a bonafide record work carried out under our guidance and supervision.

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1. Introduction

In this work, we study about differential equation of non-integer order arising in biological problems. During the past some decades, differential equation of arbitrary order have frequently appeared and become the centre of many studies, scientific and technological fields, for instance, viscoelasticity, fluid mechanics, biology, control theory, physics, electrical network, circuit system, heat transfer, social system, management, financial system, signal processing etc. In mathematical modelling, the main advantage of utilizing differential equations of fractional order is its nonlocal nature. Since the differential equation of classical order is a local in nature while the differential equation of non-integer order is non-local. Fractional calculus is the advancement of ordinary calculus by introducing derivatives of real, complex and arbitrary order. Due to many real world applications, fractional calculus has become very popular (1999, 1969, 2001, 2000, 1974). Fractional calculus is a branch of mathematics where we learn how to differentiate and integrate of an arbitrary order. In basic calculus we have learned the concept of differentiation and integration. For example, the first order, differentiation of $f(z) = z^3$ is $\frac{df(z)}{dz} = 3z^2$ and differentiating the same function to the second order result in to $\frac{d^2}{dz^2} f(z) = 6z.$ similarly $\int f(z)\, dz = \frac{1}{4} z^4 + c$ and $\int [\int f(z)\, dz]\, dz = \frac{1}{20} z^5$. Now if we want to integrate our function $f(z)$ to the $1/2\, th$ order, or find its $1/2\, th$ order derivative? How could we calculate our function? Through ordinary calculus it is very typical to get a result of this function. But we may obtain the result of this function through fractional calculus in easy and better way. Differential equations of arbitrary order are the equations that include non-integer derivatives. Arbitrary order system is system described by differential equation of non-integer order. Fractional calculus is the advancement of ordinary calculus by introducing derivatives in addition integrals of fractional order. Up to the middle of the 20th century many work has been carried in the field of special branch namely fractional calculus by famous mathematicians as, Caputo (1967), Lutzen and liouville (1982), Miller and Ross(1993) etc. The well-known books by Oldham and Spanier (1974), Nishimoto (1990), Samko et. al (1993), on fractional calculus focus on its development and use in different scientific and technological field. The applications of fractional calculus have been studied by many famous scientist and mathematicians. So name a few, Bagley and Torvik (1983,1986) discussed a model of arbitrary order of viscoelastic nature, fractionally integrated generalized

autoregressive conditional heteroskedasticity new class processes analyzed by Baillie (1996), few fundamental problems into continuum in addition statistical mechanics of arbitrary nature studies by Mainardi (1997), application of non-integer order derivative and integrals in bioengineering studied by Magin (2004), dynamics of fractional nature in interfaces represented by Chow (2005), cancer model is given by Ahmed et.al (2012), radio frequency ablation for varicose veins discussed Choi et.al (2014), Singh et.al(2018) examined for computer viruses of epidemiological model. The Drinfeld-Sokolov-Wilson equation is investigated by Singh et.al (2018), and a diabetes model of fractional nature with exponential law is analyzed by Singh et.al (2018). Here, we discuss about some standard fractional derivatives.

Fractional derivative in the Riemann-Liouville (1999) type is represented as

$$ {}_aD_X{}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-\omega)^{n-\alpha-1} f(\omega)d\omega, (n-1 \le \alpha < n). \quad (1) $$

It is generally called the Riemann-Liouville definition.

Fractional derivative in Caputo's sense (1969) is given as

$$ {}_a^cD_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(\omega)d\omega}{(x-\omega)^{\alpha+1-n}}, (n-1 \le \alpha < n). \quad (2) $$

It is usually known as Caputo's fractional derivative.

Fractional order derivative in Caputo –Fabrizio's (2015) kind is specified as follows

$$ D_x^{\alpha}(f(x)) = \frac{M(\alpha)}{1-\alpha}\int_a^x f'(\omega)\exp\left[-\alpha\frac{x-\omega}{1-\alpha}\right]d\omega. \quad (3) $$

It is generally known as Caputo- Fabrizio arbitrary order derivative.

Non-integer order derivative in Atangana –Baleanu derivative (2016, 2018) type is stated in the following way

$$ {}_a^{ABR}D_x^{\alpha}(f(x)) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dx}\int_a^x f(\omega)E_{\alpha}\left[-\alpha\frac{(x-\omega)^{\alpha}}{1-\alpha}\right]d\omega. \quad (4) $$

and $$ {}_a^{ABC}D_x^{\alpha}(f(x)) = \frac{B(\alpha)}{1-\alpha}\int_a^x f'(\omega)E_{\alpha}\left[-\alpha\frac{(x-\omega)^{\alpha}}{1-\alpha}\right]d\omega. \quad (5) $$

The equation (4) and (5) is known as Remainn –liouville sense and caputo sense respectively.

Fractional biological models are solved by various methods, such as homotopy perturbation method (HPM) [He (1999, 2000, 2006)], homotopy perturbation transform method (HPTM) [Kumar et.al (2013)], homotopy analysis transform method (HATM) [Singh (2018), Singh et. al (2018), Kumar et.al (2019)], *q*-homotopy analysis transform method (*q*-HATM) [Kumar

et. al (2017), Singh et. al (2018)]. The HPM was developed by He (1999, 2000, 2006). This technique has been employed by many researches to solve distinct fractional differential equations. A combination form of the Laplace transform technique with HPM is discussed for solving linear fractional differential equation (LFDE) and non linear fractional differential equation (NFDE), this method is known as HPTM (2011).

In real world problems, LFDE and NLFDE are solved by HATM. The HATM is an effective coupling of well known homotopy analysis method (HAM) and Laplace transform method (LTM) (2014). Strong and very effective technique *q*-HATM is a mixture of *q*-HAM (2013, 2013) in addition LTM.

2. Literature review

Mathematics has been used in biology since the 12^{th} century. For describing the increasing number of rabbits Fibonacci utilizes the famous Fibonacci sequence. By using mathematics in the 18^{th} century, the impact to smallpox on human population has been analyzed by Daniel Bernoulli. The growth of the human population was based on the exponential growth discussed by Thomas Malthus in 1789. In 1836, a logistics growth model has been developed by Pierre Francois Verhulst. Mullerian mimicry was developed by Fritz Muller which explains the evolutionary advantages in 1879. In 1917, Ronald Fisher and D'Arcy Thompson, considered one of the ground breaking texts resources may increase arithmetically. Many mathematicians are working in biological problems. Bonyah et.al (2017) analyzed a theoretical model for Zika virus. Smoking dynamics model given by Singh et.al (2017). A new technique for fractional model of human interpersonal relationships is introduced by Singh et.al (2017). Bonyah et.al (2018) studied a diarrhoea model associated with saturated incidence rate. Veeresha et.al (2019) analyzed smoking epidemic replica of fractional order by HATM. Noeiaghdam and Ghiasi (2019) discussed a strong mathematical model for solving the model of HIV virus for $CD8^+$ T-cells. We will study about use of above define methods in biological problems. These mathematical models are explained with the help of linear and non linear first order differential equations. Balamuralitharam and Geethamalini (2018) studied the HPM procedure to estimate the infection in equine infectious anemia virus model. Senthamarai and Balamuralitharm (2016) investigated the HPM to construct the approximate analytic solution of the simultaneous nonlinear ordinary differential equations arise in the model and with the help of SIRS-SI model with treatments are given to humans and mosquitoes. Khan et.al (2012) investigated the $CD4^+$ T cells HIV infection by an efficient

iterated HPTM. Kumar et.al (2013) analyzed HATM algorithm for solving generalized biological population model.

3. Description of topic

Computational methods for fractional differential equation arising in biological problems, is our topic of study. In which, we will study the biological models by applying the analytic and numerical techniques.

4. Objectives of research

The proposed research work objectives are as follows.

- To study LFDE and NLFDE arising in biological problems such as fractional tuberculosis model, fractional cancer model, fractional Ebola virus model, fractional smoking dynamic model, fractional dengue model, fractional diabetes model fractional $CD4^+$ T cells HIV infection, fractional malaria model, fractional diarrhea infection model, fractional Zika virus model etc.
- To develop analytic and numerical techniques for solving LFDE and NLFDE those arise in biological problems such as fractional tuberculosis model, fractional cancer model, fractional Ebola virus model, fractional smoking dynamic model, fractional dengue model, fractional diabetes model fractional $CD4^+$ T cells HIV infection, fractional malaria model, fractional diarrhea infection model, fractional Zika virus model etc.
- To study HPM, HPTM, HATM, *q*-HATM etc. methods to solve LFDE and NLFDE accruing in biological problems such as fractional tuberculosis model, fractional cancer model, fractional Ebola virus model, fractional smoking dynamic model, fractional dengue model, fractional diabetes model fractional $CD4^+$ T cells HIV infection, fractional malaria model, fractional diarrhea infection model, fractional Zika virus model etc.

5. Research methodology to be adopted

To solve a differential equation having fractional order derivative, we will consider the reliable numerical and analytical techniques. We will propose new methods to solve fractional differential equation. We study methods as HPM [He (1999, 2000, 2006)], the HPM was developed by He (1999, 2000, 2006). This method is a mixture of homotopy and perturbation. This technique has been employed by many researches to solve distinct

fractional differential equations. HPTM [Kumar et.al (2013)] is an amalgamation of the Laplace transform technique with the HPM which is applied for solving linear fractional differential equation (LFDE) and non linear fractional differential equation (NFDE), this method is known as HPTM. HATM [Singh (2018), Singh et.al (2018), Kumar et.al (2019)], the HATM is an effective coupling of well known homotopy analysis method (HAM) and Laplace transform method (LTM) (2014). *q*-HATM [Kumar et. al (2017), Singh et. al (2018)]. *q*-HATM is a strong and very effective technique for solving LFDE and NFDE. This method is a mixture of *q*-HAM (2013, 2013) in addition LTM. We shall also use the software such as MATLAB/MAPLE/MATHEMATICA for numerical simulations and to describe the graphical behaviour of biological models.

6. Expected outcome of the research

The proposed study is aimed to give following expected outcomes:

- The study of fractional differential equation will provide very efficient solution of biological models such as fractional tuberculosis model, fractional cancer model, fractional Ebola virus model, fractional smoking dynamic model, fractional dengue model, fractional diabetes model fractional $CD4^+$ T cells HIV infection, fractional malaria model, fractional diarrhea infection model, fractional Zika virus model etc.
- To propose novel numerical and analytical techniques to solve differential equation of fractional order arising in biological problems such as fractional tuberculosis model, fractional cancer model, fractional Ebola virus model, fractional smoking dynamic model, fractional dengue model, fractional diabetes model fractional $CD4^+$ T cells HIV infection, fractional malaria model, fractional diarrhea infection model, fractional Zika virus model etc.
- The results obtained will be very useful for medical practice, doctors, and give new directions in the field of biological problems such as fractional tuberculosis model, fractional cancer model, fractional Ebola virus model, fractional smoking dynamic model, fractional dengue model, fractional diabetes model fractional $CD4^+$ T cells HIV infection, fractional malaria model, fractional diarrhea infection model, fractional Zika virus model etc.
- New research will provide important and useful knowledge to solve differential equations of non-integer order for biological problems such as fractional tuberculosis model, fractional cancer model, fractional Ebola virus model, fractional smoking dynamic model,

fractional dengue model, fractional diabetes model fractional $CD4^+$ T cells HIV infection, fractional malaria model, fractional diarrhea infection model, fractional Zika virus model etc.

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